Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 876–887
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

First integrals and families of symmetric periodic motions of a reversible mechanical system☆

V.N. Tkhai

Moscow, Russia

Received 4 October 2005

Abstract

A reversible mechanical system which allows of first integrals is studied. It is established that, for symmetric motions, the constants of the asymmetric integrals are equal to zero. The form of the integrals of a reversible linear periodic system corresponding to zero characteristic exponents and the structure of the corresponding Jordan Boxes are investigated. A theorem on the non-existence of an additional first integral and a theorem on the structural stabilities of having a symmetric periodic motion (SPM) are proved for a system with m symmetric and k asymmetric integrals. The dependence of the period of a SPM on the constants of the integrals is investigated. Results of the oscillations of a quasilinear system in degenerate cases are presented. Degeneracy and the principal resonance: bifurcation with the disappearance of the SPM and the birth of two asymmetric cycles, are investigated. A heavy rigid body with a single fixed point is studied as the application of the results obtained. The Euler-Poisson equations are used. In the general case, the energy integral and the geometric integral are symmetric while the angular momentum integral turns out to be asymmetric. In the special case, when the centre of gravity of the body lies in the principal plane of the ellipsoid of inertia, all three classical integrals become symmetric. It is ascertained here that any SPM of a body contains four zero characteristic exponents, of which two are simple and two form a Jordan Box. In typical situation, the remaining two characteristic exponents are not equal to zero. All of the above enables one to speak of an SPM belonging to a two-parameter family and the absence of an additional first integral. It is established that a body also executes a pendulum motion in the case when the centre of gravity is close to the principal plane of the ellipsoid of inertia.

1. Formulation of the problem

Consider the reversible mechanical system[1]

$$\dot{u} = U(u, \upsilon), \quad \dot{\upsilon} = V(u, \upsilon)$$
$$U(u, -\upsilon) = -U(u, \upsilon), \quad V(u, -\upsilon) = V(u, \upsilon); \quad u \in R^l, \quad \upsilon \in R^n \ (l \geq n) \tag{1.1}$$

Here $M = \{u, v : v = 0\}$ is called the fixed set of system (1.1). The motion $u(u^0, t)$, $v(u^0, t)$ with an initial point $u^0 \in M$ is symmetric with respect to M and the conditions[2]

$$\upsilon_s(u_1^0, \ldots, u_l^0, T) = 0, \quad s = 1, \ldots, n \tag{1.2}$$

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 6, pp. 977–989, 2006.
E-mail address: tkhai@ipu.rssi.ru.

0021-8928/$ – see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2007.01.007

are necessary and sufficient for the existence of symmetric periodic motions (SPMs) of period $2T$. It can be seen that the SPMs form q-families where, in typical case, we have $q=l-n+1$.[2] The $l-n$ quantities from the initial values $u_1^0, \dots u_l^0$ plus the semiperiod T can serve as the parameters of this family.

A reversible mechanical system can allow of a first integral W. Then, at a level $W=h$(const), we have a reduced system containing the parameter h, and the q-family of SPMs in this system will be the $(q+1)$-family of SPMs in the initial system: the case which is degenerate in the general theory becomes typical. Even such a cursory glance at the problem shows the need to investigate the SPMs of a reversible mechanical system which allows of first integrals. A more detailed analysis leads to the separation of the integrals into symmetric and asymmetric integrals. It clarifies the role of each of the types of integrals and the numbers of them in the dimension of the family of SPMs, it shows the "selectness" of one of the integrals, the energy integral, when taking SPMs as the example, it gives a negative result in the problem of the existence of an additional first integral (including in a problem which is far from being an integrable problem) and it solves the problem of the continuation of an SPM with respect to a parameter. The basic result is that a typical situation has been picked out in which the conclusions are solely dictated by the number of integrals of both types.

The formulation of this problem is not only of interest in the case of a system with a first integral but, also, in the case when it is possible to construct a system which is approximate in a certain sense and allows of a first integral. We also point out that the formulation of the problem is natural for a system consisting of weakly connected subsystems.

Finally, we note that the case when $q=l-n+1$ is structurally stable[2] in the problem of the continuation of a SPM of a reversible mechanical system with respect to a parameter. Non-structurally stable cases of the theory have been considered.[2] However, the case of the existence of a first integral in the system has not been specially investigated.

Later, a heavy rigid body with a single fixed point is presented as an example of a reversible mechanical system in the typical situation.

2. First integrals and symmetric solutions in a reversible mechanical system

Definition 1. The first integral $W(u, v) = h$ (const) of system (1.1) is said to be symmetric if $W(u, -v) = W(u, v)$, and asymmetric if $W(u, -v) = -W(u, v)$.

Theorem 1. *If system (1.1) allows of a first integral of general form, then this integral is the sum of symmetric and asymmetric first integrals.*

Proof. It is obvious that the first integral of system (1.1) is always represented in the form of the sum of two functions

$$W(u, v) = F(u, v) + G(u, v); \quad F(u, -v) = F(u, v), \quad G(u, -v) = -G(u, v)$$

By virtue of the equations of system (1.1), on calculating the total derivative of the function W we obtain

$$\frac{dW}{dt} = \frac{\partial F(u, v)}{\partial u}U(u, v) + \frac{\partial F(u, v)}{\partial v}V(u, v) + \frac{\partial G(u, v)}{\partial u}U(u, v) + \frac{\partial F(u, v)}{\partial v}V(u, v)$$

Here, the first two terms are given by odd functions of v and the remaining two terms are given by even functions of v, and the total derivative of the function W is identically equal to zero. Hence, the total derivatives of the functions F and G are equal to zero. The existence of symmetric first integrals F and asymmetric first integrals G and the formula $W=F+G$ is proved by this.

Suppose system (1.1) allows of m symmetric integrals $F_\alpha(\alpha = 1, \dots, ri)$ and k asymmetric integrals $G_\beta(\beta = 1, \dots, k$. □

Theorem 2. *For symmetric motions, the constants of the asymmetric integrals are equal to zero.*

Proof. For symmetric motions, we have

$$u(-t) = u(t), \quad v(-t) = -v(t)$$

and therefore

$$G_\beta(u(-t), v(-t)) = G_\beta(u(t), -v(t)) = -G_\beta(u(t), v(t)) = 0 \quad \square$$

Corollary. *Symmetric motions separate out subspaces of dimension $l+n-k$.*

Assertion 1. If, at a point $(u^*, 0) \in M$, we have

$$\operatorname{rank}(dF_1, \ldots, dF_m, dG_1, \ldots, dG_k) = m+k$$

then $\operatorname{rank}(dF_1, \ldots, dF_m) = m$ and

$$\mathrm{Ra} \overset{\mathrm{def}}{=} \operatorname{rank}(dG_1, \ldots, dG_k) = k$$

In fact, at a point $(u^*, 0) \in M$, we have

$$\frac{\partial F_\alpha}{\partial u_1}\delta u_1 + \ldots + \frac{\partial F_\alpha}{\partial u_l}\delta u_l + \frac{\partial F_\alpha}{\partial \upsilon_1}\delta \upsilon_1 + \ldots + \frac{\partial F_\alpha}{\partial \upsilon_n}\delta \upsilon_n = \delta h_\alpha$$

$$\frac{\partial G_\beta}{\partial u_1}\delta u_1 + \ldots + \frac{\partial G_\beta}{\partial u_l}\delta u_l + \frac{\partial G_\beta}{\partial \upsilon_1}\delta \upsilon_1 + \ldots + \frac{\partial G_\beta}{\partial \upsilon_n}\delta \upsilon_n = \delta h_\beta$$

(h_α and h_β are the constants of the integrals). The functions F_α are odd with respect to υ, and therefore all $\partial F_\alpha/\partial \upsilon_i = 0$ and $\operatorname{rank}(dF_1, \ldots, dF_m) = m$. It also follows from this that $\mathrm{Ra} = k$.

Assertion 2. The condition $V(u^*, 0) \neq 0$ is a necessary and sufficient condition for a symmetric solution which differs from a constant to pass through the point $(u^*, 0) \in M$.

In fact, at the point $(u^*, 0) \in M$, we have $U(u^*, 0) \neq 0$, and the condition $V(u^*, 0) \neq 0$ is the necessary and sufficient condition for the existence of an equilibrium of a reversible mechanical system. In the case when $V(u^*, 0) \neq 0$, a symmetric solution passes through a point $(u^*, 0) \in M$.

3. Integrals of a reversible linear periodic system

We now consider the reversible linear 2π-periodic system[2]

$$\dot{x} = A^-(t)x + A^+(t)y, \quad \dot{y} = B^+(t)x + B^-(t)y; \quad x \in R^l, \quad y \in R^n \ (l \geq n) \tag{3.1}$$

Henceforth, a plus (minus) superscript denotes 2π-periodic matrices, vectors and functions composed of even (odd) functions.

System (3.1) is simultaneously invariant with respect to the two transforms

$$1)\ (x, y, t) \to (x, -y, -t), \quad 2)\ (x, y, t) \to (-x, y, -t)$$

Hence, the system has two fixed sets

$$M_x = \{x, y : y = 0\}, \quad M_y = \{x, y : x = 0\}$$

We denote the fundamental system of solutions with a unit matrix of the initial conditions by

$$S(t) = \left\| \begin{matrix} u^+(t) & u^-(t) \\ \upsilon^-(t) & \upsilon^+(t) \end{matrix} \right\|, \quad S(0) = I_{l+n} \tag{3.2}$$

(I_j is the identity $j \times j$-matrix). The condition $\upsilon^-(\pi) = n - \lambda$ then gives $l-n+k$ periodic solutions of system (3.1) which are symmetric with respect to the set M_x, and the condition $\operatorname{rank} u^-(\pi) = n - \nu$ gives ν similar solutions which are symmetric with respect to the set M_y.[2] It is obvious that some of the above-mentioned solutions can be simultaneously symmetric with respect to both of the sets M_x and M_y. The number of such solutions is equal to $\min(\lambda, \nu)$. Zero characteristic exponents correspond to periodic solutions.

By means of the substitution

$$\begin{aligned}\xi_i &= (p_i^+, x) + (q_i^-, y), \quad i = 1, \dots l\\ \eta_j &= (p_j^-, x) + (q_j^+, y), \quad j = 1, \dots n\end{aligned} \tag{3.3}$$

(the vectors $p_s^{\pm}$, $q_s^{\pm}$ are 2π-periodic with respect to t), we reduce system (3.1) to a system with constant coefficients. Here, the sets M_x and M_y pass into the sets M_ξ and M_η respectively: $M_\xi = \{\xi, \eta: \eta = 0\}$, $M_\eta = \{\xi, \eta: \xi = 0\}$.

We will now write out the part of the reduced system which corresponds to the zero characteristic exponents. We have

$$\dot{\xi}_i = 0 \; (i = 1, \dots, l-n+\min(\lambda, \nu)), \quad \dot{\eta}_j = 0, \quad j = 1, \dots, \min(\lambda, \nu) \tag{3.4}$$

$$\dot{\xi}_s = 0, \quad \dot{\eta}_s = \xi_s, \quad s = l-n+\nu+1, \dots, l-n+\lambda, \quad \nu \le \lambda \tag{3.5}$$

$$\dot{\xi}_s = \eta_s, \quad \dot{\eta}_s = 0, \quad s = l-n+\lambda+1, \dots, l-n+\nu, \quad \nu > \lambda \tag{3.6}$$

The next assertion follows from the transforms (3.3) and Eqs. (3.4)–(3.6).

Assertion 3. Suppose that, in the matrix $S(t)$ (3.2),

$$\operatorname{rank} \upsilon^-(\pi) = n - \lambda, \quad \operatorname{rank} u^-(\pi) = n - \nu$$

Then, system (3.1) contains $l - n + 2\min(\lambda, \nu) + 2|\lambda - \nu|$ zero characteristic exponents, including $|\lambda - \nu|$ pairs, each of which forms a Jordan Box.

Linear integrals correspond to each Jordan Boxwhen $\nu \le \lambda$

$$W \equiv (\alpha^+(t), x) + (\beta^-(t), y) = \text{const} \tag{3.7}$$

$$tW + (\alpha^-(t), x) + (\beta^+(t), y) = \text{const} \tag{3.8}$$

when $\nu > \lambda$

$$W_* \equiv (\alpha_*^-(t), x) + (\beta_*^+(t), y) = \text{const} \tag{3.9}$$

$$tW_* + (\alpha_*^+(t), x) + (\beta_*^-(t), y) = \text{const} \tag{3.10}$$

The integral (3.7) or (3.9) also corresponds to a simple characteristic exponent.

4. The dependence of the period of a SPM on the constants of the integrals

Consider the q-family of SPMs

$$u = \varphi(h, t), \quad \upsilon = \psi(h, t)$$

where a motion with a semiperiod $T(h^*) = \pi$ corresponds to the values $h_1^*, \dots, h_q^*$.

The functions $\varphi(h, (T/\pi)t)$, $\psi(h, (T/\pi)t)$ have a period equal to 2π, which is independent of the parameter h. The same is true for their derivatives with respect to h_j. We now calculate these derivatives, labelling the substitution $h = h^*$ with a subscript asterisk:

$$p_j(t) = \left(\frac{\partial\varphi}{\partial h_j}\right)_* + \frac{t}{\pi}\left(\frac{\partial T}{\partial h_j}\right)_*\left(\frac{\partial\varphi}{\partial t}\right)_*, \quad g_j(t) = \left(\frac{\partial\psi}{\partial h_j}\right)_* + \frac{t}{\pi}\left(\frac{\partial T}{\partial h_j}\right)_*\left(\frac{\partial\psi}{\partial t}\right)_* \tag{4.1}$$

The functions $(\partial\varphi/\partial h_j)_*$ and $(\partial\psi/\partial h_j)_*$ constitute a system of q independent solutions of the system of variational Eq. (3.1). These solutions are symmetric with respect to the set M_x.

It can be seen from equalities (4.1) that, when $(\partial T/\partial h_j)_* = 0$, we have a periodic solution and, conversely, for a periodic solution, we have $(\partial T/\partial h_j)_* = 0$.

System (3.1) always has $l-n$ periodic solutions which are symmetric with respect to the set M_x and it is therefore necessary that $l-n$ partial derivatives of T with respect to h_j should vanish.

We will now consider the case when k asymmetric integrals exist in system (1.1)

Lemma. *If an SPM with a semiperiod T passes through a point $(u^*, 0) \in M$, $u^* = u(0)$, and $\mathrm{Ra} = k$ then (1) $\mathrm{Ra} = k$ also at the point $(u^{**}, 0) \in M$, $u^{**} = u(T)$, (2) k equations in system (1.2) are corollaries of the remaining equations, (3) system (1.1) has an r-family of SPMs $(r \geq l-n+k)$ with a fixed period of 2π, (4) the variational system (3.1) admits of $k^* \geq k$ periodic solutions which are symmetric with respect to M_y and $r^* \geq r$ periodic solutions which are symmetric with respect to M_x.*

Proof. We linearize the asymmetric integrals for the SPM being considered and then arrive at k integrals of the form (3.9). According to Assertion 3, integral (3.9) corresponds to a periodic solution of the variational system (3.1), and the solution is symmetric with respect to the set M_y. Since the number k of such solutions is the same at the points $(u^*, 0)$, $(u^{**}, 0) \in M$, the magnitude of Ra is the same at these points. □

We consider Eq. (1.2) at the point $(u^{**}, 0)$ together with the equalities

$$G(u(u^0, T), v(u^0, T)) = 0$$

Then, the condition $\mathrm{Ra} = k$ enables us to analyse a system of just $n-k$ equations instead of system (1.2), which contains n equations: the remaining k equations are automatically satisfied. As before, this reduced system contains l unknowns u_j^0 and a parameter T but its solution now depends on no less than $l-n+k$ parameters and T, which signifies the existence of an r-family of SPMs $(r \geq l-n+k)$ with a fixed period of $2T$. In system (3.1), this leads to $r^* \geq r$ periodic solutions which are symmetric with respect to M_x.

Theorem 3. *If the reversible mechanical system (1.1) has k asymmetric first integrals, then the partial derivatives of the period of the SPM with respect to the $l-n+k$ parameters h_j of the family are equal to zero.*

Proof. According to the lemma, the system has an r-family of SPMs $(r \geq l-n+k)$ with a fixed period of $2T$ and the linear system (3.1) possesses $r^* \geq r$ periodic solutions which are symmetric with respect to the set M_x. We therefore obtain from formulae (4.1) that $l-n+k$ derivatives $\partial T/\partial h_j$ are equal to zero. □

Corollary. *If $q = l-n+k+1$ in system (1.1), then, at the point h^*, we have*

$$dT = a\,dh_q, \quad a = \text{const}$$

that is, in the first approximation, the period of an SPM only depends on a single parameter.

Remark. For Theorem 3 to hold, it is not obligatory that symmetric integrals exist.

5. A typical situation for the SPMs of a reversible mechanical system

The natural condition $\mathrm{rank}(dG_1, \ldots, dG_k) = k$ means that the system of asymmetric integrals is non-degenerate at a point $(u^*, 0) \in M$ through which an SPM with a period of 2π passes. When this condition is satisfied, the lemma guarantees the existence in Eq. (3.1) of $k^* \geq k$ periodic solutions which are symmetric with respect to M_y.

The number of parameters r of a family of SPMs with a fixed period is no less than $l-n+k$. Suppose the vector parameter $h = (h_*, h_{**})$ of a family of SPMs contains two components, and $\partial T/\partial h_* = 0$, $\partial T/\partial h_{**} \neq 0$. According to Theorem 3, the dimension $\dim h_* \geq l-n+k$. Here, the strict inequality arises in the degenerate case when one of the derivatives which is non-zero vanishes.

Suppose the parameters of a reversible mechanical system are specified. Then, degeneracy is revealed in the phase space when an observer moves in it along a family of SPMs. Another reason for the occurrence of an atypical situation for SPMs is a change in the parameters of the system.

The number of integrals of the form (3.9) is equal to k^*. This means that, in the case when $k^* > k$, the linear system (3.1) has more asymmetric integrals than the overall system. This situation also has to be recognized as being atypical of SPMs.

Definition 2. We will say that a situation in which the number k^* of integrals of the form of (3.9) is identical to the number k of asymmetric first integrals and $\dim h_* = l - n + k$ is typical for SPMs.

In the typical situation, we obtain k simple characteristic exponents corresponding to linear integrals of the form (3.9). The remaining simple characteristic exponents correspond to integrals of the form (3.7). On the other hand, the variational system (3.1) always his a periodic solution ($\dot{\varphi}$, $\dot{\psi}$) which is symmetric with respect to M_y, and we therefore obtain a single Jordan box. In the general case, there can be several such boxes. According to relations (3.5), each such box gives an increasing solution which is symmetric with respect to M_x.

We will now formulate conclusions which until now have been explicitly associated with the existence of symmetric integrals.

Theorem 4. *In the typical situation, we have*

$$\mathrm{Ra}_1 \overset{\mathrm{def}}{=} \mathrm{rank}\left\|\frac{\partial \upsilon_s(u_1^0, \ldots, u_l^0, T)}{\partial u_j^0}\right\|_* = n - k$$

(*an asterisk denotes substitution of the values* $u^0 = u^*$, $T = T^* = \pi$).

Proof. It follows from the lemma that, in the typical situation, Eq. (3.1) only has $l - n + k$ periodic motions which are symmetric with respect to the set M_x; Jordan boxes do not give such solutions. This means that $\mathrm{Ra}_1 = n - k$. □

Theorem 5. *In the typical situation, the dimension q of the family of SPMs is equal to $l - n + k + 1$ and it necessarily includes the $(l - n + k)$-subfamily of motions with a fixed period.*

Proof. Because of the existence of SPMs, the reduced system (1.2) based on k asymmetric integrals admits of a solution: $u^0 = u^*$, $T = \pi$. According to Theorem 4, we have $\mathrm{Ra}_1 = n - k$, and the implicit function theorem enables us to find the solution of a reduced system containing $l - n + k + 1$ arbitrary parameters, one of which is the parameter T, uniquely. The solution when $T = T^*$ gives a subfamily with a fixed period $2T$. □

Corollary. *In the typical situation,* $\dim h_{**} = 1$.

We will now find the structurally stable cases in the theory of the continuation of an SPM with respect to a parameter in the case of a reversible mechanical system.

Theorem 6. *The following holds in the typical situation regardless of the form of the actual perturbations: (a) in a perturbed autonomous system, a family of SPMs is continued with respect to a parameter in the cases when $k = 0$ and $k = 1$, (b) in the case when $k = 0$, an $(l - n)$-family of SPMs is created under the action of periodic perturbations.*

Proof. In the case when $k = 0$, we have $\mathrm{Ra}_1 = n$, and the question of the existence of SPMs when there are perturbations is solved by the application of the implicit function theorem to system (1.2). In the case when $k = 1$, we have $\mathrm{Ra}_1 = n - 1$ but

$$\mathrm{Ra}_2 \overset{\mathrm{def}}{=} \mathrm{rank}\left\|\frac{\partial \upsilon_s(u^0, T)}{\partial u_j^0}, \frac{\partial \upsilon_s(u^0, T)}{\partial T}\right\|_* = n - k$$

from where we obtain the second part of assertion a. □

The existence of symmetric integrals has not been specifically mentioned in Theorems 4–6 above. We will assume that system (1.1) admits of m symmetric integrals. The arbitrary constants of these integrals can then be chosen as the parameters of the family of SPMs.

Theorem 7. *In the typical situation an SPM contains $l - n + 2k$ simple zero characteristic exponents and* max$\{1, m - (l - n + k)\}$ *Jordan boxes corresponding to the zero characteristic exponents.*

Proof. Suppose $m \leq l - n + k$, that is, the dimension of the subfamily of SPMs with a fixed period is no less than the number of symmetric integrals. Here, in any case we have $l - n + k$ periodic solutions which are symmetric with respect to M_x (and $l - n + k$ integrals of the form (3.7)) and k such solutions which are symmetric with respect to M_y

(and k integrals of the form of (3.9)). In addition to this, we necessarily have a single Jordan box which corresponds to the solution $(\dot\varphi, \dot\psi)$. Hence, the number of zero characteristic exponents for the SPM is equal to $l-n+2k+2$ and there is just a single Jordan box.

Suppose $m>l-n+k$. The dimension of the family of SPMs with a fixed period is equal to $l-n+k$ and system (3.1) only has k periodic solutions which are symmetric with respect to M_y. This means, firstly, that the family of SPMs is independent of $m-(l-n+k+1)$ constant integrals and, secondly, that we obtain $m-(l-n+k)$ Jordan boxes including the box with the solution $(\dot\varphi, \dot\psi)$. □

Remark. It follows from Theorem 7 that, when $m=l-n+k+1$, just a single integral is picked out from among all of the symmetric integrals and, in the examples, this is the energy integral.

The following assertion is associated with the non-existence of additional first integrals in a reversible mechanical system.

The question of the existence of an additional integral, which differs from the energy integral, in Hamiltonian systems was raised for the first time by Poincaré.[3] The negative answer to this question is associated with the production of isolated oscillation at a specified energy level.[4] In a reversible mechanical system, this fact follows from the existence of the typical family of SPMs (in the form of oscillations and/or rotations).

Theorem 8. *If it is known that a reversible mechanical system admits of m symmetric and k asymmetric first integrals, where $m>l-n+k$ and the SPM of this system contains $2m-(l-n)$ zero characteristic exponents, then, in the typical situation, there is no first integral in the system in addition to the known first integrals.*

Proof. In the typical situation, all of the zero characteristic exponents are given by Theorem 7 and there are therefore no other zero characteristic exponents in the system corresponding to additional first integrals. □

Corollary. *In a reversible mechanical system which depends on parameters, it is necessary to seek additional first integrals in a set of parameters of zero measure where an atypical situation is realized for the SPM.*

Remark.

1°. The condition $m>l-n+k$ is a natural condition and is satisfied, for example, in the three-body problem ($m=1$, $l=n=2$, $k=0$), in the case of a heavy rigid body with a single fixed point (see Section 6), etc.
2°. In the case when $m \le l-n+k$, the missing first integrals, if there are any, can only be symmetric.
3°. The assertion is well known in the case of a Hamiltonian system. The Poincaré method gives an answer in the case of systems that are close to being integrable.[4]

6. A heavy rigid body with a single fixed point

The motion of a heavy rigid body with a single fixed point is described by the Euler-Poisson equations

$$\begin{aligned} A\dot p &= (B-C)qr + P(z_0\gamma_2 - y_0\gamma_3), \quad \dot\gamma_1 = \gamma_2 r - \gamma_3 q \\ B\dot q &= (C-A)rp + P(x_0\gamma_3 - z_0\gamma_1), \quad \dot\gamma_2 = \gamma_3 p - \gamma_1 r \\ C\dot r &= (A-B)pq + P(y_0\gamma_1 - x_0\gamma_2), \quad \dot\gamma_3 = \gamma_1 q - \gamma_2 p \end{aligned} \tag{6.1}$$

Here A, B and C are the principal moments of inertia of the body, P is the weight of the body, x_0, y_0, z_0 are the coordinates of the centre of gravity, $\omega=(p, q, r)$ is the angular velocity and $\gamma=(\gamma_1, \gamma_2, \gamma_3)$ is the unit upwards vector.

System (6.1) admits of the classical integrals

$$W_1 = Ap^2 + Bq^2 + Cr^2 + 2P(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = 2h(\text{const})$$

$$W_2 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \sigma(\text{const})$$

$$W_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

A characteristic feature of system (6.1) is its invariance with respect to the replacement R: $(\omega, \gamma, t) \to (-\omega, \gamma, -t)$. This means the system (6.1) belongs[5] to the class of reversible mechanical systems with a fixed set $M = \{\omega, \gamma: \omega = 0\}$. The energy and geometric integrals are symmetric with respect to M, that is,

$$W_{1,3}(-p, -q, -r, \gamma_1, \gamma_2, \gamma_3) = W_{1,3}(p, q, r, \gamma_1, \gamma_2, \gamma_3)$$

while the angular momentum integral turns out to be asymmetric

$$W_2(-p, -q, -r, \gamma_1, \gamma_2, \gamma_3) = -W_2(p, q, r, \gamma_1, \gamma_2, \gamma_3)$$

In the case when the centre of gravity is located in the principal plane of the inertia ellipsoid ($y_0 = 0$), system (6.1) is also invariant with respect to the replacement

$$R_y : (p, q, r, \gamma_1, \gamma_2, \gamma_3, t) \to (p, -q, r, \gamma_1, -\gamma_2, \gamma_3, -t)$$

that is, it allows of a second fixed set

$$M_y = \{p, q, r, \gamma_1, \gamma_2, \gamma_3 : q = 0, \gamma_2 = 0\}$$

In this case, all the classical integrals become symmetric with respect to the fixed set M_y, that is,

$$W_j(p, -q, r, \gamma_1, -\gamma_2, \gamma_3) = W_j(p, q, r, \gamma_1, \gamma_2, \gamma_3)$$

Almost all known exact solutions of Euler's problem, apart from permanent rotations, belong[6] to the case when $y_0 = 0$. For example, Mlodzeyevskii pendulum motions,[7] regular Grioli precessions,[8] etc. are symmetric with respect to the set M_y. It is found[9,10] that the Grioli solutions belong to the two-parameter family of SPMs.

In the case when $y_0 = 0$, the Euler-Poisson Eq. (6.1) can be written in the form of the reversible system (1.1) with $l = 4$, $n = 2$ and the vectors $u = (p, q, \gamma_1, \gamma_3)^T$, $v(q, \gamma_2)^T$.

Theorem 9. *For a body with its centre of gravity in the principal plane of the ellipsoid of inertia ($y_0 = 0$) in the typical situation, the SPM of system (6.1) contains two simple zero characteristic exponents, a single pair of zero characteristic exponents which form a Jordan box, and the remaining two characteristic exponents are calculated by constructing just a single solution of the Cauchy problem.*

Proof. Of this follows from the ratio of the dimensions of the vectors u and $v (l - n = 2)$, the existence of three symmetric integrals, Theorem 7 and the method of calculating the characteristic exponents of a reversible system in Ref. 9. □

Remark. In the typical case, two of the characteristic exponents are non-zero.

Theorem 10. *For arbitrary specified parameters A, B and C, x_0, z_0 and $y_0 = 0$ in the typical situation, the SPMs of system (6.1) form a two-parameter family depending on h and σ, containing a subfamily of specified period which depends on the parameter σ. This family is continued (when $y_0 = 0$) with respect to the parameters of the problem.*

Proof. This assertion follows from Theorems 5 and 6, the existence of three symmetric first integrals and taking account of the fixed constant in the geometric integral. Then, two simple zero characteristic exponents are associated with the angular momentum integral and the geometric integral; the integrals give the parameter σ. The other parameter h is supplied by the energy integral and the pair of zero characteristic exponents with a Jordan box which are associated with it. □

Remark. Theorem 1 enables us to extend the previously described observation[9,10] to any SPM of the problem.

We will now analyse two remarkable SPMs in system (6.1).

6.1. Regular Grioli precessions

Grioli[8] discovered regular precessions in the case of a body fixed at a point such that the conditions

$$x_0^2(B-C) = z_0^2(A-B), \quad y_0 = 0, \quad A > B > C \tag{6.2}$$

are satisfied. Grioli precessions have two remarkable features: (1) they arise during the motion of a dynamically asymmetric system which is solely subjected to conditions (6.2), (2) the body precesses about an axis inclined to the vertical axis at a certain angle β, zather than about the vertical axis.

It can be seen from the explicit formulae for Grioli solutions[11] that, under the fixing conditions (6.2), we have a mechanically unique possible motion in the form of a precession which is a SPM.[9] Nevertheless, we obtain from Theorem 10 that the given SPM belongs to an h and σ two-parameter family and this family is continued in the case when the first condition of (6.2) is approximately satisfied.

6.2. Mlodzeyevskii pendulum motions

These motions occur in a problem when $y_0 = 0$ without any additional constraints on the moments of inertia and the suspension point and contain SPMs both in the form of oscillations as well as in the form of rotations:

$$\begin{aligned} &B\dot{q} = P(x_0\gamma_3 - z_0\gamma_1), \quad \dot{\gamma}_1 = -q\gamma_3, \quad \dot{\gamma}_3 = q\gamma_1 \\ &p = r = 0, \quad \gamma_2 = 0 \end{aligned} \tag{6.3}$$

The family of SPMs (6.3) is parametrized by a natural parameter, that is, by the constant of the energy integral h. The characteristic exponents for solution (6.3) have been calculated in Ref. 12: the typical case is obtained. Consequently, the one-parameter Mlodzeyevskii family belongs (Theorem 10) to the two-parameter family of SPMs, containing not only plane motions but also motions close to plane ones.

It is noteworthy that pendulum oscillations are simultaneously symmetric with respect to the two fixed sets: M and M_y. The symmetry of the oscillations with respect to M enables us to apply Theorem 6 to the general case ($y_0 \neq 0$) of problem (6.1) and to obtain the following result.

Theorem 11. *In problem (6.1) with a centre of gravity close to the principal plane of the ellipsoid of inertia, a one-parameter, h, family of SPM oscillations which are close to plane ones, always exists.*

Proof. In the case when $y_0 \neq 0$ in the reversible mechanical system (6.1), we have $l = n = 3$, $m = 2$ and $k = 1$. In the typical situation, according to Theorem 5 we have a one-parameter family of SPMs (the constant in the geometric integral is fixed). □

Definition 3. The one-parameter family the energy constant h being the parameter of symmetric periodic motions, connecting the upper and the lower equilibrium positions, is called the pendulum oscillations of a heavy rigid body with a single fixed point.

It follows from Theorem 11 that pendulum oscillations are the most general SPMs of the problem. These motions occur both in the case when $y_0 = 0$ (Mlodzeyevskii oscillations) as well as in the general case. In the Euler-Poinsot case, the above mentioned oscillations degenerate into equilibrium positions. In the remaining classical cases of integrability, these SPMs contain six zero characteristic exponents.

Theorem 12. *In the typical situation, problem (6.1) does not have a first integral in addition to the classical first integrals.*

Proof. Pendulum oscillations are the most general SPMs of a body, and the proof therefore follows from Theorem 8. □

Remark. In the typical situation, the pendulum oscillations contain, as in the case when $y_0 = 0$, two simple zero characteristic exponents plus two zero characteristic exponents forming a Jordan box and two non-zero characteristic exponents of opposite sign.

Theorem 12 is well known[4] for cases which are close to integrable (also, see Refs. 13–16). The problem of the non-existence of an additional integral for other cases remained uninvestigated. The integral found in Ref. 17 in a closely related problem indicates the interest in this problem. It follows from the characteristic of the typical situation, in which set of parameters additional first integrals can be sought.

Some of the results in Section 6 were announced in Ref. 18.

7. A quasilinear system

On passing into the neighbourhood of an SPM, we obtain a problem on the investigation of a quasilinear system. Taking a second order system as an example, below we present results on oscillations in degenerate cases, which were obtained using of the general theoretical results presented earlier in Refs. 2,19.

Consider the family of systems

$$\dot{u} = \xi v + \mu U_1(\mu, u, v, t), \quad \dot{v} = \eta u + \mu V_1(\mu, u, v, t)$$

when $\xi = 0, 1$, $\eta = -1, 0, 1$.

7.1. *The case when $\xi = 0$ and $\eta = 1$*

When $\mu = 0$, we have a family of equilibria on the v-axis ($u = 0$). When $\mu \neq 0$ an isolated SPM (periodic perturbations) or an equilibrium (autonomous system) arise from the point $u = 0$, $v = 0$.

7.2. *The case when $\xi = 1$ and $\eta = -1$*

This case is non-structurally stable. In the generating system, we find isochronous oscillations which are symmetric with respect to the sets $M_1 = \{u, v : v = 0\}$ and $M_2 = \{u, v : u = 0\}$. When $\mu \neq 0$, we change to variable amplitude and angle: A, θ ($u = A\cos\theta$, $v = A\sin\theta$). Then,

$$\dot{A} = \mu(U_1\cos\theta + V_1\sin\theta), \quad \dot{\theta} = -1 + \frac{\mu}{A}(-U_1\sin\theta + V_1\cos\theta)$$

It can be seen that, for small $\mu \neq 0$, the angle θ changes monotonically and, in the case of an autonomous system, we therefore obtain a family of SPMs which is close to the family of isochronous oscillations. The period of the motion serves as the parameter of the family.

In the case of a periodic system, we make use of the amplitude equation[19]

$$\int_0^{\pi}[-U_1(0, A\cos t, -A\sin t, t)\sin t + V_1(0, A\cos t, -A\sin t, t)\cos t]dt = 0$$

the simple root $A = A^*$ of which guarantees the existence of an isolated SPM.

7.3. *The case when $\xi = \eta = 0$*

Here, the simple root $u^0 = u^*$ of the amplitude equation

$$\int_0^{\pi} V_1(0, u^*, 0, t)dt = 0 \tag{7.1}$$

ensures[19] the existence of an isolated SPM. We note that, in the case of time-dependent perturbations U_1 and V_1, the cases $\xi = 1$, $\eta = -1$ and $\xi = \eta = 0$ reduce to one another.

7.4. *The case when $\xi = 1$ and $\eta = 0$*

When $\mu = 0$, we have a family of equilibria lying on the u-axis. In the case when $\mu \neq 0$, the simple root of Eq. (7.1) guarantees the existence of an SPM in the form of a cycle.

8. Ordinary and critical points of the family of SPMs. Principal resonance

In the family of SPMs, the period $T(h)$ depends on the parameter h, where

$$\dim h = q, \quad h = (h_*, h_{**}), \quad \partial T/\partial h_* = 0, \quad \partial T/\partial h_{**} \neq 0$$

In the typical situation, we have $\dim h_* = l - n + k$, $\dim h_{**} = 1$.

Definition 4. The point h of a family of SPMs is said to be an ordinary point if $\dim h_*(h) = l - n + k$ and a critical point if $\dim h_*(h) > l - n + k$.

Theorem 6 on the structural stability of a property to have SPMs is proved for ordinary points. In the simplest case of system (1.1), when $l = n = 1$, a periodic perturbation at an ordinary point always initiates the birth of a cycle which differs from the generating SPM for the magnitude of the perturbation.[20] Here, the case when $\xi = 0$ and $\eta = 1$ (Section 7) arises on passing into the neighbourhood of an SPM.

Below, we consider system (1.1) when $l = n = 1, m \geq 0, k = 0, q = 0$ and show which scenario of oscillations is possible in the case of a critical point.

It is obvious that, on passing in the phase space from one SPM to another of a family, the derivative T'_h can vanish: a critical point arises. In a linear system, $dT(h) \equiv 0$. The critical points in a non-linear system are an exception. The rule is the existence of ordinary points.

At a critical point, solution (4.1) is periodic, and an analogue of the case when $\xi = 1$ and $\eta = -1$ (Section 7) is realized. A principal resonance arises here under the action of 2π-periodic perturbations μU_1 and μV_1.

We now pass into the neighbourhood of an SPM and put

$$u = \varphi(h^*, t) + x, \quad \upsilon = \psi(h^*, t) + y$$

We make use of a normal form which is continuous with respect to μ.[21] In the complex-conjugate variables w and $\bar{w}$ (the required second order system is separated out), we have

$$\dot{w} = i(C_{20}w^2 + C_{11}w\bar{w} + C_{02}\bar{w}^2) + i\mu a_0$$

(C_{kj} and a_0 are real coefficients). Now, after the conversion

$$w = \mu r(\cos\theta + i\sin\theta)$$

we obtain

$$\begin{aligned} \dot{r} &= \mu^{1/2}(C_{-}\sin\theta + C_{11}\sin 3\theta)r^2 \\ \dot{\theta} &= \mu^{1/2}[(C_{+}\cos\theta + C_{11}\cos 3\theta)r + a_0/r] \\ &(C_{\pm} = C_{20} \pm C_{02}) \end{aligned} \tag{8.1}$$

System (8.1) admits of a family of SPMs. The same holds for the system in the variables x and y, and the equality

$$C_{+} + C_{11} = 0 \tag{8.2}$$

must therefore be satisfied in the normal form (8.1).The system of amplitude equations[19] is derived here from the conditions

$$\dot{r} = 0, \quad \dot{\theta} = 0$$

It can be seen that the system does not have roots for which $\sin\theta = 0$. On the other hand, because of equality (8.2), we find two simple roots $(r_0, \pm\theta_0)$

$$r_0^2 = \frac{a_0}{2(C_{02} - C_{20})\cos\theta_0}, \quad 1 - 2\sin^2\theta_0 = \frac{C_{02}}{C_{20}}, \quad \sin\theta_0 \neq 0 \tag{8.3}$$

The following result therefore holds.

Theorem 13. *A principal resonance arises at a critical point of a family of SPMs. Bifurcation occurs here: the SPM disappears but two asymmetric cycles (8.3) arise, and an amplitude of the oscillations is of the order of* $\mu^{1/2}$.

Acknowledgements

I wish to thank V. V. Rumyantsev for his interest.

This research was supported financially by the Russian Foundation for Basic Research (06-01-00068) and the programme for the State Support for Leading Scientific Schools (NSh-6667.2006.1)).

References

1. Tkhai VN. Reversible mechanical systems. In: *Non-linear Mechanics*. Moscow: Fizmatlit; 2001. p. 131–46.
2. Tkhai VN. The continuation of the periodic motions of a reversible system in non- structurally stable cases. Application to an N-planet problem. *Prikl Mat Mekh* 1998;**62**(1):56–72.
3. Poincare H. *Les methods nouvelles de la mecanique celeste*, Vol. 1. Paris: Gauthier-Villars; 1892.
4. Arnold VI, Kozlov VV, Neishtadt AI. *Mathematical Aspects of Classical and Celestial Mechanics*. Moscow: Editorial URSS; 2002.
5. Tkhai VN. Reversibility of mechanical systems. *Prikl Mat Mekh* 1991;**55**(4):578–86.
6. Gorr GV, Kudryashova LA, Stepanova LA. *Classical Problems in Rigid Body Dynamics. The development and present state*. Kiev: Naukova Dumka; 1978.
7. Mlodzeyevskii BK. On permanent axes in the motion of a heavy body around a fixed point. *Tr Otd Fiz Nauk Obshch Lyubit Yetestv Antropol Etnorg* 1894;**7**(1):46–8.
8. Grioli G. Esistensa e determinazione della precessioni regolari dinamicamente possibili per un solido pesante assimetrico. *Ann Mat Pura Ed Appl* 1947. Ser. 4. Vol. 26. Facs. 3–4. 271–281.
9. Tkhai VN. The stability of regular Grioli precessions. *Prikl Mat Mekh* 2000;**64**(5):848–57.
10. Tkhai VN. Periodic motions of a heavy rigid body with a single fixed point close to regular Grioli precessions. In: *Problems in Research on the Stability and Stabilization of Motion.* Moscow: Vychisl Tsentr Ross Akad Nauk; 2000; **1**: 60–67.
11. Gulyayev MP. A new particular solution of the equations of motion of a heavy rigid body with a fixed point. *Vestnik MGU Ser Fiz-Mat Yetestv Nauk* 1955;**3**:15–21.
12. Tkhai VN, Shvygin AL. Stability of rotations around a horizontal axis of a heavy rigid body with a single fixed point. In: *Problems in Research on the Stability and Stabilization of Motion.* Moscow: Vychisl Tsentr Ross Akad Nauk 2000; **2**:149–157.
13. Kozlov VV. *Methods of Qualitative Analysis in Rigid-body Dynamics*. Moscow: Izd MGU; 1980.
14. Ziglin SL. Branching of the solutions and the non-existence of first integrals in Hamiltonian mechanics. *Funkts Analiz Yego Prim* 1982;**16**(3):30–41;
    Ziglin SL. Branching of the solutions and the non-existence of first integrals in Hamiltonian mechanics. *Funkts Analiz Yego Prim* 1983;**17**(1):8–23.
15. Kozlov VV, Treshchev DV. The Non-integrability of a general problem on the rotation of a dynamically symmetric heavy rigid body with a fixed point. *Vestnik MGU Ser 1 Matematika Mekhanika* 1985;**6**:73–81;
    Kozlov VV, Treshchev DV. The Non-integrability of a general problem on the rotation of a dynamically symmetric heavy rigid body with a fixed point. *Vestnik MGU Ser 1 Matematika Mekhanika* 1986;**1**:39–44.
16. Dovbysh SA. Splitting of the separatrices of unstable uniform rotations and the non-integrability of the perturbed Lagrange problem. *Vestnik MGU Ser 1 Matematika Mekhanika* 1990;**3**:70–7.
17. Sokolov VV. A new integrable case for Kirchhoff's equation. *Teor Mat Fiz* 2001;**129**(1):31–7.
18. Tkhai VN. Families of symmetric periodic motions in Euler's problem. *Dokl Akad Nauk* 2005;**401**(4):483–5.
19. Tkhai VN. The Lyapunov-Poincaré method in the theory of periodic motions. *Prikl Mat Mekh* 1998;**62**(3):335–71.
20. Tkhai VN. Periodic motions of a second-order reversible mechanical system. Application to the Sitnikov problem. *Prikl Mat Mekh* 2006;**70**(5):813–34.
21. Bryuno AD. *The Local Method of Non-linear Analysis of Differential Equations*. Moscow: Nauka; 1979.

Translated by E. L. S.